Five Year Integrated M.Sc. Examination, 2024
Semester-V
Subject: Analysis-I
Course Code: MT-3-5-2

Time: 4 Hours **Full Marks: 80**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Attempt any four questions.

1. Find out the correct answer from the following.
(a) Consider the limit $\lim_{\epsilon \to 0} \dfrac{1}{\epsilon} \displaystyle\int_0^\infty e^{-x/\epsilon}(\cos(3x) + x^2 + \sqrt{x+4})$.
(i) The limit does not exist. (ii) The limit exists and is equal to 0. (iii) The limit exists and is equal to 3. (iv) The limit exists and is equal to π.
(b) Let $\alpha = \int_0^\infty \dfrac{1}{1+t^2} dt$, then
(i) $\dfrac{d\alpha}{dt} = \dfrac{1}{1+t^2}$ (ii) α is a rational number (iii) $\log(\alpha) = 1$ (iv) $\sin(\alpha) = 1$
(c) Consider the improper Riemann integral $\int_0^x y^{-\frac{1}{2}} dy$. This integral is
(i) continuous in $[0, \infty)$ (ii) continuous only in $(0, \infty)$ (iii) discontinuous in $(0, \infty)$ (iv) discontinuous only in $(0.5, \infty)$
(d) Consider the improper integral $I = \int_1^2 \frac{\sqrt{x}}{\log x} dx$ then
(i) I is convergent (ii) I is divergent (iii) $I = 0$ (iv) $I = 1$
(e) If $f : [a, b] \to R$ is continuous then
(i) f is integrable on R (ii) f is integrable on $R - [a, b]$ (iii) f must be integrable on $[a, b]$ (iv) f is need not integrable on $[a, b]$.
(f) If the function $f : R \to R$ is defined as $f(x) = [x]$, then
(i) f(x) is continuous on R (ii) f(x) is differentaible on R (iii)f(x) is Reimann integrable on R (iv) (i)f(x) is not Riemann integrable on R
(g) If $f(x) = 1/q$ for any rational number x and 0 otherwise, then the lower Riemann integral is
(i) 6 (ii) 2 (iii) 0 (iv) 1 3+3+3+3+3+3+2=20

2. (a) If $f(x)$ be bounded on the closed interval [a, b], then show that

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a).$$

(b) State and prove Darboux's theorem.

(c) Given $f(x) = 0, x$ is rational and $f(x) = 1, x$ is irrational. Prove that $f \notin \Re[a, b]$ for any $a < b$.
(d) If $f(x)$ be continuous on $[a, b]$, then $f \in \Re[a, b]$. 4+6+4+6=20

3. (a) For the integral $\int_0^1 x dx$, find the upper and lower integral sums corresponding to the division [0, 1] into 3 and 6 equal intervals.

(b) Show that $[x]$ is integrable on [0,3] and $\int_0^3 [x] dx = 3$.
(c) Show that $\frac{1}{2} < \displaystyle\int_0^1 \frac{dx}{\sqrt{4 - x^2 + x^3}} < \frac{\pi}{6}$.
(d) State and prove first mean value theorem for integrals. 6+4+4+6=20

4. (a) Test the convergence of $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!} + ...$ by Cauchy criterion.

(b) Prove that $\{f_n(x)\} = \left\{\dfrac{x}{nx + 1}\right\}$ converges uniformly to 0 on [0, 1].
(c) Show that if $\{f_n\}$, $n \in N$ be a sequence of continuous functions on an interval I, and if $f_n \to f$ uniformly on I, then f is continuous on I.
(d) Let $\{f_n(x)\} = nxe^{-nx^2}$, $n \in N$ and $x \in [0, 1]$. Determine whether

$$\lim_{n \to \infty} \int_0^1 f_n(x) dx = \int_0^1 \{\lim_{n \to \infty} f_n(x)\} dx.$$

6+4+4+6=20

5. (a) State and prove the Cauchy -Hadamard formula for the power series $\sum a_n x^n$.
(b) Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ and discuss the convergence at each end of the interval.
(c) Starting from the power series expansion of $(1 - x^2)^{-\frac{1}{2}}$, derive the power series of $\sin^{-1} x$. Hence obtain the sum of the series $1 + \frac{1}{2} . \frac{1}{3} + \frac{1.3}{2.4} \frac{1}{5} + \frac{1.3.5}{2.4.6} . \frac{1}{7} + ...$
(d) Find the series for $\log(1 + x)$ by integration and use Abel's theorem to show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$ =log 2. 6+4+6+4=20

6. (a) Does the improper integral $\displaystyle\int_0^\infty \frac{1}{1 + x^2} dx$ exist?
(b) Assuming the integrals to be convergent show that

$$\int_0^{\frac{\pi}{2}} \log \sin x dx = \int_0^{\frac{\pi}{2}} \log \cos x dx = \frac{\pi}{2} \log \frac{1}{2}.$$

(c) Show that $\displaystyle\int_2^\infty \frac{dx}{x \log x}$ diverges.

(d) Show that $\int_1^\infty \frac{\sin x}{x^p} dx$ converges absolutely for $p > 1$ but only conditionally if $0 < p \leq 1$.

4+7+4+5=20